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On the generalized relativistic Liouville equilibrium†

E Alvarez
Departamento de Física Teórica, Universidad Autónoma de Madrid, Canto Blanco, Madrid, Spain

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Abstract. We discuss the possibilities for the introduction of a scalar entropy $S(x)$ in the framework of the relativistic kinetic theory, in order to obtain a generalized 'equilibrium' distribution function when the cross section is identically zero ('Liouville equilibrium'). We further study its particular form in uniform space–times, its properties against those of the Jüttner–Synge function and the meaning of its non-relativistic limit.

1. Introduction

It is well known that a collision-free relativistic distribution function $f(x, p)$ must satisfy the 'one-particle Liouville equation',

$$\mathscr{L}(\boldsymbol{X})f(x, p)=0 \tag{1}$$

where $\mathscr{L}(\boldsymbol{X})$ denotes the Lie derivative with respect to $\boldsymbol{X}$, and $\boldsymbol{X}$ is the vector field in the particle phase space $P_7(V_4)$ defined by

$$\boldsymbol{X}\equiv(p^{\alpha}, Q^{j}\equiv-P^{j}_{\mu\nu}p^{\mu}p^{\nu})$$

where

$$\alpha, \beta=0, 1, 2, 3 \qquad i, j\ldots=1, 2, 3.$$

Equation (1) of course implies that f is a constant of motion. The problem of finding $f(x, p)$ reduces then to the problem of constructing constants of motion and choosing a functional form for the function.

In this paper we shall make use of a physical interpretation of the zeroth order moment of the distribution function proposed by Bel (1971, private communication) and Choquet-Bruhat (1971, unpublished) to justify a particular functional form for $f(x, p)$. In § 3 we show that this function is compatible with the metric of uniform model universes, and can be used as a 'global equilibrium' function in the kinetic theory of cosmology. As is well known, the Boltzmann equilibrium (Jüttner–Synge) function is not compatible with non-stationary space–times; thus it can only be used in cosmology to describe 'local equilibrium' (Hakim 1968).

In § 4 we compare the non-relativistic limits of the Jüttner–Synge and our 'generalized' functions and show that the limit of the first is the classic Maxwell–Boltzmann function and the limit of the second is a new distribution function, the

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solution of the classic Liouville equation for the dynamic system corresponding to the Newtonian cosmology.

2. The scalar entropy density

Bel (1969, 1971, private communication) and Choquet-Bruhat (1971, unpublished) have shown that the zeroth-order moment of the distribution function

$$A(x) \equiv \int_{P(x)} f(x, p)\omega(p)$$

where $P(x)$ is the mass hyperboloid and $\omega(p)$ is the invariant volume element on $P(x)$

$$P(x) \equiv \{\boldsymbol{p} \in Tx, \boldsymbol{p}^2 = -m^2\} \qquad \omega(p) \equiv \sqrt{|g|}\frac{dp^1 \wedge dp^2 \wedge dp^3}{|g_{0\alpha}p^\alpha|}$$

gives, when the particle mass m is not zero†, a physical interpretation. Given a domain G of the space–time manifold V_4, the integral

$$A \equiv \int_G \eta A(x)$$

(where η is the volume element on V_4: $\eta \equiv \sqrt{|g|}\,dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$) represents the sum of the proper times during which each trajectory of a particle remains in the considered region. In the case $m \neq 0$, the distribution function is then able to be normalized from the zeroth-order moment, and we can interpret it as an 'occupation number of proper times granules'. We shall then define the action density as $A(x)$ and the action entropy as

$$S(x) \equiv -\int_{P(x)} f \ln f\omega(p). \tag{2}$$

The well known probability argument (Synge 1957)‡ states that the probability of a given distribution depends monotonically on $S(x)$. As a condition to be satisfied by a Liouville 'equilibrium' distribution function we shall maximize $S(x)$, while keeping invariant the first two moments which are $N^\alpha(x)$, the particle current vector, and $T^{\alpha\beta}(x)$, the energy–momentum tensor. We begin with the 'variational' principle

$$\delta S(x) + \lambda_\alpha \delta N^\alpha(x) + \mu_{\alpha\beta}\delta T^{\alpha\beta}(x) = 0 \tag{3}$$

whose solutions (Bel 1971, private communication) are of the form

$$f(x, p) = B(x)\exp(\lambda_\alpha(x)p^\alpha + \mu_{\alpha\beta}(x)p^\alpha p^\beta). \tag{4}$$

If we want the moments of f to converge, $\boldsymbol{\lambda}$ must be timelike, i.e. $\boldsymbol{\lambda}^2 \leq 0$ (we use signature +2) and $\mu_{\alpha\beta}p^\alpha p^\beta \leq 0$.

The condition of equation (4) gives the functional form of an 'equilibrium' distribution. Perhaps the name 'generalized equilibrium distribution function' is justified, since when $\mu_{\alpha\beta} \equiv 0$ we re-obtain the Jüttner–Synge distribution function.

† We shall assume that this mass is constant and the same for all particles of the gas.
‡ See appendix for a detailed discussion.

Supposing now that equation (4) verifies the Liouville equation (1), we obtain (u^α being the unitary tangent vector to a geodesic of V_4 which represents the mean four-velocity of the fluid)

$$u^\alpha \nabla_\alpha (\mu_{\alpha\beta} p^\alpha p^\beta) = 0 \qquad u^\alpha \nabla_\alpha (\lambda_\rho p^\rho) = 0 \qquad u^\alpha \nabla_\alpha B = 0 \tag{5}$$

that is, B must be a constant and $\lambda_\rho p^\rho$ and $\mu_{\alpha\beta} p^\alpha p^\beta$ first integrals of the geodesic equations of V_4. We shall suppose, in view of future cosmological applications, that $\lambda_\rho \equiv 0$, i.e. that V_4 is non-stationary.

3. The generalized equilibrium distribution function

In order to deduce some properties of the 'Bel function'

$$f(x, p) = B \exp(\mu_{\alpha\beta} p^\alpha p^\beta) \tag{5a}$$

we shall work in local orthonormal coordinates (Ehlers 1971) in which

$$\omega(p) = (E^2 - m^2)^{1/2} \, \mathrm{d}E \wedge \sin\theta \, \mathrm{d}\theta \wedge \mathrm{d}\phi.$$

Defining

$$\mu_{\alpha\beta} p^\alpha p^\beta \equiv -z m^{-2} (E^2 - m^2) \tag{5b}$$

which will be justified later, we find for the action density

$$A = \pi\sqrt{\pi} B m^2 U(\tfrac{3}{2}, 2, z) \tag{6}$$

where $U(a, b, z)$ is the Kummer function (Magnus *et al* 1966). If we now break down the first moments of the distribution function in the normal way in terms of the basic tensors $g_{\alpha\beta}$ and u_α, we get

$$N^\alpha \equiv n u^\alpha \equiv \frac{\rho}{m} u^\alpha \qquad T^{\alpha\beta} \equiv (\mu + p) u^\alpha u^\beta + p g^{\alpha\beta}.$$

For the 'numerical world density' n, 'energy density' μ and 'isotropic pressure' p, we obtain expressions:

$$n = \pi\sqrt{\pi} B m^3 z^{-3/2} \qquad \mu = \pi\sqrt{\pi} B m^4 U(\tfrac{3}{2}, 3, z) \tag{7}$$
$$3p = \pi\sqrt{\pi} B m^4 (U(\tfrac{3}{2}, 3, z) - U(\tfrac{3}{2}, 2, z)).$$

These equations obviously imply that the 'perfect gas equation of state' $p = nkT$, which is satisfied by the Jüttner–Synge function, does not hold for our Bel function. We can, however, obtain 'energetic' equations, relating p and μ in both the low ($z \to \infty$) and high ($z \to 0$) temperature limits.

In fact, using the well known formulae (cf for example Magnus *et al* 1966)

$$U(a, b, z)_{|z| \to 0} = \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b} + \mathrm{O}(|z|^{\mathrm{Re}\, b - 2}) \qquad \mathrm{Re}\, b \geqslant 2, b \neq 2$$

$$U(a, b, z)_{|z| \to 0} = \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b} + \mathrm{O}(|\ln z|) \qquad b = 2$$

$$U(a, b, z)_{|z| \to \infty} = z^{-a} \left(\sum_{u=0}^{R-1} \frac{(a)_n (1 + a - b)_n}{n!} (-z)^{-n} + \mathrm{O}(|z|^{-R}) \right)$$

we obtain

$$\mu = \rho \qquad (|z| \to \infty) \qquad\qquad \mu = 3p \qquad (|z| \to 0) \tag{8}$$

which coincides with the corresponding formulae for the Jüttner–Synge function when $T \to 0$ and $T \to \infty$, respectively. Moreover, in these two limits both the Jüttner–Synge and the Bel functions can be represented by a perfect fluid with equation of state $p \sim n^\gamma$ where $\gamma = \frac{4}{3}$ (ultra-relativistic, $T \to \infty$) and $\gamma = \frac{5}{3}$ (classic, $T \to 0$). This type of result could lead us to suppose that the physical description obtained from both functions must be very similar to the two limits under consideration. It is at intermediate temperatures—in a physical sense—that we can expect differences. The preliminary results of Alvarez and Gracia (1973, unpublished) point in this direction.

4. Bel's function in uniform space–times

It is well-known (Ehlers *et al* 1968) that the Robertson–Walker spaces, with the following metric in coordinates adapted to the mean velocity $\boldsymbol{u}$,

$$ds^2 = dt^2 + R(t)^2 K(r)^2 \delta_{ij}\, dx^i\, dx^j \tag{9}$$

admit a first integral of second order to the geodesic equations, i.e.,

$$q^2 \equiv R(t)^2 h_{\alpha\beta} p^\alpha p^\beta \equiv R(t)^2 (g_{\alpha\beta} + u_\alpha u_\beta) p^\alpha p^\beta. \tag{10}$$

We can also write the metric equation (9) in coordinates adapted to $\boldsymbol{\xi}$, the generator of conformal rigidness (Bel 1969):

$$ds^2 = -\xi^2 (dx^0)^2 + \xi^{-2} K^2 \delta_{ij}\, dx^i\, dx^j \tag{11}$$

where $\xi = \xi(x^0)$ is a function of time only. Our first integral is†

$$q^2 = \xi^{-2} h_{\alpha\beta} p^\alpha p^\beta \tag{12}$$

and the 'Bel function' equation (5), writing $\mu_{\alpha\beta} p^\alpha p^\beta \equiv -\zeta^2 q^2$ (ζ a real constant), becomes

$$f(x, p) = B \exp(-\zeta^2 q^2). \tag{13}$$

The mean four-velocity of the relativistic gas at a point P of the space–time manifold V_4 is given by

$$\boldsymbol{u}(P) = \xi^{-1} \boldsymbol{\xi}(P). \tag{13a}$$

The relative spatial velocity of a particle with four-momentum $\boldsymbol{p}$, with respect to the mean $\boldsymbol{u}$ is

$$v_\alpha \equiv -\xi (u_\rho p^\rho)^{-1} h_{\alpha\mu} p^\mu \tag{13b}$$

which implies

$$v^2 = g_{\alpha\beta} v^\alpha v^\beta = \xi^2 (u^\rho p_\rho)^{-2} h_{\alpha\beta} p^\alpha p^\beta \qquad q = mv\xi^{-2}(1 - v^2 \xi^{-2})^{-1/2} \tag{14}$$

$$v = q\xi^2 (m^2 + \xi^2 q^2)^{-1/2}. \tag{15}$$

† This form of the first integral justifies the assumption made in writing (5b), choosing $z \equiv m^2 \zeta^2 \xi^{-2}$.

Thus the first integral q is related to the relative velocity v, and becomes zero only when $v=0$. The limit of equation (15) when $q\to\infty$ is ξ. This fact is related to the 'conformal' (Bel and Escard 1966, Alvarez and Bel 1973) interpretation of the space metric, in which the light velocity (with $c = 1$) is v (light) $= \xi$. In the Minkowskian limit of equation (11) (i.e., $\xi\to 1$, $K\to 1$), the first integral equation (14) has the limit

$$q^2\to m^2\gamma^2v^2 \qquad \gamma\equiv(1-v^2)^{-1/2}.$$

Making a series development in v^n (low velocities),

$$q^2=m^2(v^2+O(v^4))$$

we can then write at $v^4=0$ order,

$$f_{\rm Bel}\to B\exp(-m^2\zeta^2v^2). \tag{16}$$

Identifying equation (16) with the Maxwell–Boltzmann distribution function, we are led to the interpretation

$$\zeta^2=\phi(x^\rho)/2mkT \tag{17a}$$

where

$$\phi(x^\rho)|_{\rm Minkowski}\equiv 1.$$

In the next section we shall show that there is another possible limit of equation (13), without the 'weak field' assumption implicit in equation (17a).

When $\zeta\to\infty$ (i.e. $T\to 0$), from equation (7) (with $z\equiv m^2\zeta^2\xi^{-2}$ in our case) and the asymptotic expressions of the Kummer functions, we obtain

$$\frac{p}{n}=\frac{\xi^2}{2m\zeta^2}\left(1-\frac{5}{4}\frac{\xi^2}{m^2\zeta^2}+O(\zeta^{-4})\right)$$

which, upon substituting equation (17a), reduces to

$$\frac{p}{n}=\frac{\xi^2kT}{\phi(x)}\left(1-\frac{5}{2}\frac{\xi^2kT}{m\phi}+O(T^2)\right). \tag{18a}$$

Identifying equation (18a) at the $T^2=0$ limit with the perfect gas equation $p=nkT$, we get $\phi(x)=\xi^2$, i.e.

$$\zeta^2=\xi^2/2mkT \tag{17b}$$

$$\frac{p}{n}=kT\left(1-\frac{5}{2}\frac{kT}{m}+O(T^2)\right) \tag{18b}$$

so we obtain the usual equation of state only when $T^2=0$.

5. The non-relativistic limit

In the last section we have seen that the Minkowskian, low-velocity limit of the Bel function is (as in the Jüttner–Synge function) the Maxwell–Boltzmann function. But in doing this, we have made the weak-field (Minkowskian) approximation which seems inadequate for describing non-static Newtonian (cosmological) situations. The Maxwell–Boltzmann distribution function (with constant temperature) is the solution of the classic one-particle Liouville equation only when v^2 is a first integral of the equation of motion; i.e. $a^i = \mathrm{d}v^i/\mathrm{d}t = 0$. When the corresponding dynamical system of R^3 has an $a^i \neq 0$, we must, as in the relativistic case, look for first integrals of the equations of motion.

The idea is then to associate a dynamical system in R^3 with the geodesic equations of V_4, written in a particular system of coordinates, in such a way that the low-velocities limit of the relativistic first integral q^2 equation (10) is a first integral of the Newtonian dynamical system.

This can be done (Alvarez and Sanz, paper in preparation) starting from the comoving coordinates of equation (9), for example, whose non-zero Christoffel symbols are (in this section the velocity of light will be denoted by c and for simplicity $K = 1$, i.e. Euclidean 3-spaces $x^0 = cte$)

$$\Gamma^0_{ij} = RR'\,\delta_{ij} \qquad \Gamma^i_{0j} = \frac{R'}{R}\delta^i_j \qquad R' \equiv \frac{\mathrm{d}R}{\mathrm{d}x^0}. \tag{19}$$

we associate these with the following geodesic equations with affine parameter

$$\frac{\mathrm{d}^2x^0}{\mathrm{d}\tau^2} \equiv \frac{\mathrm{d}u^0}{\mathrm{d}\tau} = -\Gamma^0_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} \qquad \frac{\mathrm{d}^2x^i}{\mathrm{d}\tau^2} \equiv \frac{\mathrm{d}u^i}{\mathrm{d}\tau} = -\Gamma^i_{\alpha\beta}u^\alpha u^\beta \tag{20}$$

to give a dynamical system in R^3 defined by $\mathrm{d}^2x^i/\mathrm{d}t^2 = a^i$ with $(v^i/c \equiv u^i/u^0)$:

$$\frac{a^i}{c} \equiv \frac{\mathrm{d}v^i}{\mathrm{d}x^0} \equiv \frac{\mathrm{d}\tau}{\mathrm{d}x^0}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(c\frac{u^i}{u^0}\right). \tag{21}$$

Substituting the expressions of $\mathrm{d}u^\alpha/\mathrm{d}\tau$ from equation (20) one gets

$$a^i = -\Gamma^i_{00}c^2 - 2\Gamma^i_{0j}cv^j - \Gamma^i_{jk}v^jv^k + v^i(c\Gamma^0_{00} + 2\Gamma^0_{0j}v^j + \Gamma^0_{kl}v^kv^l/c). \tag{22}$$

In comoving coordinates, introducing equation (19) gives

$$a^i = \left(-2\frac{\dot{R}}{R} + R\dot{R}\frac{v^2}{c^2}\right)v^i$$

with $\dot{R} \equiv \mathrm{d}R/\mathrm{d}t = cR'$.

The Newtonian limit of this dynamical system is

$$a^i = -2\frac{\dot{R}}{R}v^i\dagger. \tag{23}$$

It is easy to show that q^2 (Newtonian) $= R^4v^2$ is a first integral of the dynamical system equation (23) and precisely the limit of the relativistic q^2 when $c \to \infty$:

$$q^2(\mathrm{relat}) = \frac{m^2R^4v^2}{1 - v^2/c^2} \to m^2R^4v^2.$$

Here $v^2 \equiv \delta_{ij}v^iv^j$ and v^i is given by equation (13*b*).

† One can show that the dynamical system (23) is completely equivalent to the one usually used in Newtonian cosmology (for example, Harrison 1967), i.e.

$$a^i = \frac{\ddot{S}}{S}x^i, \qquad v^i = \frac{\dot{S}}{S}x^i$$

by identifying $S = R^{-2}$.

We can then say that the non-relativistic limit of the Bel function in Robertson–Walker space–times is

$$f_{\mathrm{Bel}}(\text{non-relat}) = B \exp(-mR^2v^2/2kT) \tag{24}$$

which is the solution of the classic one-particle Liouville equation for the Newtonian dynamical system (23)

$$a^i = -\frac{2\dot{R}}{R}v^i.$$

6. Conclusions

We believe that this result clearly shows the physical meaning of the Bel function: its classical limit (without the 'weak field' assumption) being the distribution function equation (24), a solution of the classical one-particle Liouville equation coupled with the dynamical system equation (23). One can, of course, interpret equation (24) as a 'local Maxwell' function, i.e., one with a temperature depending on the place according to the law $T = T_0/R^2$ and so, classically, this new function does not lead to a new 'equilibrium'. But in the general-relativistic (cosmological) situation a similar interpretation of the Bel function in terms of the Jüttner–Synge one does not seem possible, and so the physical predictions must be different in both cases, as shall be shown in a subsequent paper.

It is perhaps worth remarking that preliminary calculations carried out in the framework of the kinetic theory of cosmology (Alvarez *et al* 1975) seem to indicate that Bel functions adequately describe the cosmological gas.

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Appendix. On the scalar entropy density

It is well-known (cf Ehlers 1971) that the one-particle relativistic distribution function is usually normalized by pointing out that

$$N = \int_G \sigma_\alpha N^\alpha, \qquad \sigma_\alpha \equiv \tfrac{1}{6}\eta_{\alpha\beta\delta\gamma}\mathrm{d}x^\beta \wedge \mathrm{d}x^\gamma \wedge \mathrm{d}x^\delta, \qquad N^\alpha \equiv \int_{K_x} p^\alpha f \omega_p$$

is the average number of particles whose worldlines interest the hypersurface $G \subset V_4$ and which have, at the point x of intersection, a 4-momentum p contained in $K_x \subset P(x)$. As has been remarked by Hakim (1967), this approach is based on the fact that the zeroth order moment of the DF has no direct physical meaning, and does not correspond to a normalization of f; the DF is normalized, then, only through the first moment (the current).

A very similar calculation shows that the entropy density must be defined as a 4-vector:

$$S^\alpha(x) \equiv -\int_{K_x} f \ln f p^\alpha \omega_p$$

in order for

$$S \equiv \int_G \sigma_\alpha S^\alpha$$

to represent the mean value of $-\ln f$ on the 'instantaneous phase space' defined from $G \subset V_4$ and $K_x \subset P(x)$. From this and the relativistic Boltzmann equation, one can demonstrate the relativistic H theorem, i.e., $\nabla_\alpha S^\alpha \geqslant 0$.

In this paper, we follow an alternative approach first introduced by Bel, which is based on a physical interpretation of the zeroth-order moment of the DF as the proper time density. One can then normalize the DF from this 'action density' and introduce a scalar entropy, as has been done in the paper. The 'scalar' variational principle can then be applied as in the classical situation, but as yet no H theorem has been derived from this approach.

As the more intricate point seems to be the proposed interpretation of the zeroth order moment of the DF we outline here two different proofs of it. There exists, in fact, another one by Bel (1968) that has not yet been published.

Proof 1. Choquet-Bruhat (1971) defines proper time (setting for simplicity $m=1$) as a 1-form τ on the trajectories of the vector field $\boldsymbol{X}$, such that $i(\boldsymbol{X})\tau = 1$ (i.e., the 1-form τ, when integrated on a trajectory of $\boldsymbol{X}$, gives the proper time of the corresponding part of the worldline of the material particle).

Then, defining $\theta \equiv \eta \wedge \omega$, one can easily show that $\theta = \tau \wedge i(\boldsymbol{X})\theta$ (we recall that on surfaces $\Sigma \subset P(V_4)$ constructed from $G \subset V_4$, $K_x \subset P(x)$, $i(\boldsymbol{X})\theta$ reduces to $p^\alpha \sigma_\alpha \wedge \omega$).

It follows that it is equivalent to interpret $f\eta \wedge \omega$ as giving the measure element from the mean 'presence number' of particles in phase space $P(V_4)$, and $f\, i(\boldsymbol{X})\theta = f p^\alpha \sigma_\alpha \wedge \omega$ as the mean number of particle paths crossing a 6-submanifold. The theorem by Choquet-Bruhat show that in the first case the 'presence number' of particles in a region must be interpreted as the sum of the proper times during which each trajectory remains in the considered region.

Then, if $T \equiv \int f\eta \wedge \omega$ is the proper time, one must interpret $A \equiv \int f\omega$ as 'proper time density'.

Proof 2. One can also use the special relativistic formalism introduced by Hakim (1967). He starts from the equations of motion of a particle with mass m:

$$m\,\mathrm{d}u^\mu/\mathrm{d}s = P^\mu(x^\rho, p^\sigma) \qquad m\,\mathrm{d}x^\mu/\mathrm{d}s = p^\mu$$

whose Cauchy problem has a solution of the form

$$x^\mu(s) = x^\mu(s; x_0^\mu, p_0^\nu) \qquad p^\mu(s) = p^\mu(s; x_0^\mu, p_0^\nu)$$

such that $x^\mu(0) = x_0^\mu$ and $p^\mu(0) = p_0^\mu$.

In μ space $P(V_4)$, this trajectory can be represented by the density

$$R(x^\rho, p^\mu; s) = \delta[x^\rho - x^\rho(s; x_0^\mu, p_0^\nu)] \otimes \delta[p^\rho - p^\rho(s; x_0^\mu, p_0^\nu)]$$

such that

$$\int_{P(V_4)} R(x^\mu, p^\sigma; s)\eta \wedge \omega = 1.$$

Then, Hakim considers an ensemble (Gibbs) of similar systems (the same equations of motion) and ensures that the initial measures are distributed according to $D_0(x_0^\mu, p_0^\nu)$, such that

$$\int_{P(V_4)} D_0(x_0^\mu, p_0^\nu)\eta \wedge \omega = 1.$$

It follows that, at a given proper time s, the density in μ space is no longer $R(x^\rho, p^\rho, s)$ and we must take instead its average value

$$D(x^\rho, p^\mu; s) \equiv \langle R(x^\rho, p^\sigma; s)\rangle = \int_{P(V_4)} R(x^\mu, p^\sigma; x_0^\mu, p_0^\nu; s) D_0(x_0^\mu, p_0^\nu)\eta \wedge \omega.$$

Hakim has shown two lemmas connecting this formalism with the more usual one of the distribution functions:

(i) $$\lim_{s\to\infty} D(x^\rho, p^\sigma; s) = 0$$

(ii) $$f(x, p) = \int_{-\infty}^{+\infty} D(x^\rho, p^\sigma; s)\,\mathrm{d}s.$$

With the help of these results we can easily show the meaning of the zeroth-order moment: the number of particles within a (finite Lebesque-measurable) four-volume $G \subset V_4$, whose proper time is s, will be

$$n_G(s) = \int_G \rho(x^\mu, s)\eta$$

with the 'local instantaneous density' ρ defined by

$$\rho(x^\mu, s) = \int_{P(x)} D(x^\mu, p^\rho; s)\omega_p.$$

The total 'number' of particles in G (sum of the proper times during which the trajectories remain in G) will be (applying lemma (ii))

$$\begin{aligned} T = \int_{-\infty}^{+\infty} n_G(s)\,\mathrm{d}s &= \int_{-\infty}^{+\infty} \mathrm{d}s \int_G \eta\rho(x^\mu, s) = \int_{-\infty}^{+\infty} \mathrm{d}s \int_G \eta \int_{P(x)} \omega D(x^\rho, p^\sigma; s) \\ &= \int_G \eta \int_{P(x)} \omega \int_{-\infty}^{+\infty} \mathrm{d}s D(x^\rho, p^\mu; s) = \int_G \eta \int_{P(x)} \omega f(x, p) = \int_G \eta A(x) \end{aligned}$$

thus leading to the proposed interpretation for the zeroth-order moment.

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